Intro to course

To quote Kontsevich & Soibelman

"Rephrasing the well-known quote of Gelfand, one can say that <u>any</u> area of mathematics is a kind of <u>deformation theory</u>"

So what is "deformation theory"?

K & S say: "the study of moduli spaces of structures"

So what is a "moduli space"?

K & S seem to think everyone knows this, but I'll exhibit less confidence & at least indicate what the term means!

<u>Vague attempt 1</u>:

A moduli space $\mathcal{M}$ parametrizes some "family of structures". It should be specified by how other spaces map <u>into</u> it, since $X \xrightarrow{f} \mathcal{M}$ will then provide some X-family of structures.

<u>Vague example 1</u>:

People talk about a "moduli space of genus g Riemann surfaces", usually denoted $\mathcal{M}_g$

$\mathcal{M}_g$ is identified by the property that

$$\{ S \xrightarrow{f} \mathcal{M}_g \} \cong \left\{ \begin{array}{c} \Sigma \\ \downarrow \pi \\ S \end{array} : \begin{array}{l} \text{(flat) families over} \\ S \text{ w/ each fiber a genus} \\ g \text{ smooth curve} \end{array} \right\} \Big/ \text{iso}$$

cf classifying spaces in topology, where

$$\pi_0 \operatorname{Maps}(S, BG) \cong \{G\text{-bundles on } S\} / \text{iso}$$

Precise example 1

In algebraic geometry, every commutative ring R defines a scheme called $\operatorname{Spec} R$. (You don't need to know what a scheme is for this class!)

$$\begin{array}{ccc} \mathrm{CAlg} & \xrightarrow{\operatorname{Spec}} & \mathrm{Sch}^{op} \\ R & \longmapsto & \operatorname{Spec} R \end{array}$$

← so I will only discuss maps $\operatorname{Spec} R \to \mathcal{M}$ here

Consider the functor giving the "group of units"

$$\mathcal{M} : R \longmapsto R^\times$$

thus an R-family of points in $\mathcal{M}$ is a unit in R

N.B. $\mathcal{M} = \operatorname{Spec} \mathbb{Z}[t, t^{-1}]$ since

$$\operatorname{Hom}_{\mathrm{CRngs}}(\mathbb{Z}[t,t^{-1}], R) = R^\times \quad \forall R$$

Better attempt

If you are familiar w/ categories, you may recognize the following idea:

$$\mathcal{S} \hookrightarrow \mathrm{Fun}(\mathcal{S}^{op}, \mathrm{Set})$$ "Yoneda embedding"

some category of "spaces", e.g. schemes or htpy types

$$X \longmapsto (h_X : Y \longmapsto \mathrm{Hom}_{\mathcal{S}}(Y, X))$$

~~Alternate~~ We call h_X ~~its~~ "functor of points"

<u>Def</u> A moduli space is a space specified by its functor of points
(i.e., we give $\mathcal{M} \in \mathrm{Fun}(\mathcal{S}^{op}, \mathrm{Set})$ rather than some $X \in \mathcal{S}$ directly)

<u>Rmk</u>: It can be really cool to find multiple descriptions of the same space (e.g. "group of units" vs $\mathbb{Z}[t^{\pm 1}]$) as they each provide a distinct perspective (& tools!)

Precise example 2

Fix $k < n$ positive integers.

Let $\mathcal{M} : R \longmapsto \left\{ \underset{R\text{-submodule}}{M \subseteq R^n} : R^n/M \text{ is projective of rank } n-k \right\}$

" $=$ " "$\{$rk k submodules of $R^n\}$"

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Q: What is another name for $\mathcal{M}$?

Answer: the Grassmannian Gr_k^n

Another moduli description:

$$R \mapsto GL_n(R) \Big/ GL_k(R) \times GL_{n-k}(R)$$

Another "concrete" description:

Plücker embedding $Gr_k^n \hookrightarrow \mathbb{P}(\Lambda^k \mathbb{R}^n)$ $\lrcorner$

Now that you have an idea of "moduli space", let's try "deformation theory"

Vague attempt 1

Given a moduli space $\mathcal{M}$ & a point $p: * \longrightarrow \mathcal{M}$, ($* = \operatorname{Spec}\mathbb{C}$)
the "formal moduli problem" $\mathcal{M}_p$ is the "collection of points very close to p in $\mathcal{M}$" or its neighborhood, suitably understood

Vague attempt 2 2^{nd} half of today

Better attempt Next couple of weeks!

Vague example

Let $\mathbb{D} = \mathbb{C}[\varepsilon]/(\varepsilon^2)$ "dual numbers"

$$\begin{tikzcd} \mathbb{D} = \mathbb{C}[\varepsilon]/(\varepsilon^2) \arrow[d, "\varphi"', "\text{“}\varepsilon=0\text{”}"] \\ \mathbb{C} = \mathbb{C}[\varepsilon]/(\varepsilon) \end{tikzcd}$$

φ is a deformation of p to "first order"

then

$$M_p(\mathbb{D}) = \left\{ \begin{tikzcd} \mathrm{Spec}\,\mathbb{D} \arrow[r, "\varphi"] & M \\ \ast = \mathrm{Spec}\,\mathbb{C} \arrow[u, "\mathrm{Spec}\,\varphi"] \arrow[ur, "p"'] \arrow[u, phantom, shift left=4, "\circlearrowleft"] \end{tikzcd} \right\}$$

[also known as T_pM "tangent space"]

<u>Ex</u>

Fix $\mathbb{C}^k \overset{p}{\hookrightarrow} \mathbb{C}^n$

$(a_1,\dots,a_n) \mapsto (a_1,\dots,a_k,0,\dots,0)$

then $(Gr_k^n)_p(\mathbb{D}) \cong M_{n\times n}(\mathbb{C}) \big/ M_{k\times k}(\mathbb{C}) \times M_{n-k\times n-k}(\mathbb{C})$

matrices

Notice that this is a <u>quotient</u> of a Lie algebra by sub Lie algebras $\quad gl_k \times gl_{n-k} \hookrightarrow gl_n$

To quote K & S:

"An experimental fact (and perhaps a meta-theorem) is that for any deformation problem $\boxed{\text{in char zero}}$ one can find a dg Lie algebra "controlling" it"

Proving a precise version of this assertion (following Lurie's approach, although Hinich & Pridham others? also gave proofs) is the goal of the first 2/3 of course.

"History" & motivation

I

Like a lot of math, deformation theory grew out of mechanics & the challenges it posed to mathematics. Usually called <u>perturbation theory</u>.

<u>Problem 1</u> (classical mechanics)

2-body gravitational problem is solvable (for point particles!)

but solar system has <u>lots</u> of (non-pointlike) particles

<u>idea</u>: ~~treat~~ replace true problem by a "small perturbation" of 2-body problem & hope solutions are also "perturbed"

<u>Problem 2</u> (quantum mechanics)

Simplest abstract version of QM is the eigenproblem

"Hamiltonian" H = self-adjoint operator on a Hilbert space

solve $Hv = Ev$ ← eigenvalue is "energy"

Suppose H_0 has discrete eigendata

$$\{ (v_n^0, E_n^0) \}$$

w/ ~~[illegible]~~ $(v_m^0, v_n^0) = \delta_{mn}$ (O.N. basis)

Consider a perturbation

$$H_\varepsilon = H_0 + \varepsilon H'$$

w/ "ε small": Solve for eigendata $\{(v_n^\varepsilon, E_n^\varepsilon)\}$

Ansatz:

"solve to order n" by letting $\varepsilon^{n+1} = 0$

In math: work over base algebra $\mathbb{C}[\varepsilon]/(\varepsilon^{n+1})$

$n=1$ case (= "1st order perturbations")

$\mathbb{C}[\varepsilon]/(\varepsilon^2)$ is the dual numbers

$$(H_0 + \varepsilon H')(v_n^0 + \varepsilon v_n^1) = (E_n^0 + \varepsilon E_n^1)(v_n^0 + \varepsilon v_n^1)$$

$$\Updownarrow \quad \varepsilon^2 = 0$$

$$\underbrace{H_0 v_n^0}_{\text{cancel}} + \varepsilon\left(H_0 v_n^1 + H' v_n^0\right) = \underbrace{E_n^0 v_n^0}_{\text{cancel}} + \varepsilon\left(E_n^0 v_n^1 + E_n^1 v_n^0\right)$$

Take inner product w/ v_n^0:

$$\underbrace{(v_n^0, H_0 v_n^1)}_{\text{self-adjoint} \to \,=\, (H_0 v_n^0, v_n^1) \,=\, \underbrace{E_n^0 (v_n^0, v_n^1)}_{\text{cancel}}} + (v_n^0, H' v_n^0) = \underbrace{E_n^0 (v_n^0, v_n^1)}_{\text{cancel}} + E_n^1 \cancel{(v_n^0, v_n^0)}^{\,1}$$

$$\boxed{E_n^1 = (v_n^0, H' v_n^0)}$$

1st order change in energy
$=$
expected value of H' on v_n^0

~~The simplicity~~

This result is used all over in QM.

Rmk 2^{nd} order formula $(\varepsilon^2/\varepsilon^2)$ is also very simple

$$E_n^2 = \sum_{m \neq n} \frac{|(v_m^0, H' v_n^0)|^2}{E_n^0 - E_m^0}$$

Big picture

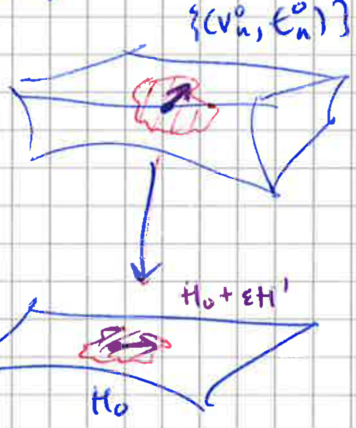
eigenbasis eigenvalues

$U(1)^N$ acts on bases ↻ moduli of "solutions" $= \left\{ \begin{matrix} \text{ordered} \\ \text{O.N. bases} \end{matrix} \right\} \times \mathbb{R}^N$

$\Big\downarrow$

moduli of "Hamiltonians" (w/ discrete spectra)

"nice" QM systems



Rmk To start, physicists are happy to work to finite order & do not worry about convergence!

(or nonanalytic families)

We would call this "formal" deformation theory.

Abstract pattern : A problem P gives:

"nilpotent" algebra

$A_1 = \mathbb{C}[\varepsilon]/(\varepsilon^4)$

or

$A_2 = \mathbb{C}[\underline{\varepsilon_1, \varepsilon_2}]/(\varepsilon_1^5, \varepsilon_2^3)$

multiple parameters

$\rightsquigarrow$ set of solutions

$\mathrm{Sol}_P(A_1)$

$\uparrow$ restricts via $\varepsilon_2 = 0$, $\varepsilon = \varepsilon_1$

$\mathrm{Sol}_P(A_2)$

In other words: there is a functor

$$\underbrace{\mathrm{Def}_P : \underset{\text{(local) Artinian algebras}}{\underline{\mathrm{Art}}} \longrightarrow \mathrm{Set}}_{\text{subject of Claudia's talk}}$$

II

Algebra (s!)

Let's turn to a standard example from algebra:

$\boxed{\text{the moduli of associative algebras}}$

Moduli

For concreteness, let $\underline{A}$ be a finite-dimensional vector space

A multiplication is a bilinear map

$$m \in \mathrm{Hom}(A \otimes A, A)$$

such that

$$m \circ (m \otimes id_A) = m \circ (id_A \otimes m)$$

$$\left[(ab)c = a(bc) \right]$$

$$\to \{d_2 = 0\} \subset \mathrm{Hom}(A^{\otimes 2}, A) \xrightarrow{d_2} \mathrm{Hom}(A^{\otimes 3}, A)$$

$$m \longmapsto m \circ (m \otimes id_A) - m \circ (id_A \otimes m)$$

$\wr$

associative products

But this variety isn't yet the moduli:

if $g \in \mathrm{Aut}(\underline{A}) = GL(\underline{A})$, then $g^{-1} \circ m \circ g = g \bullet m$

is associative $\Leftrightarrow$ m is associative

& g would be an algebra isomorphism

Hence we want

$$\{d_2 = 0\} \Big/ GL(\underline{A})$$

better to do a smarter kind of quotient but let's not focus on that now

$\boxed{\text{Deformation problem}}$

Fix $m_0 \in \{d=0\}$ & consider 1st order def:

$$m_0 + \varepsilon m \quad , \quad \varepsilon^2 = 0$$

Then

$$\begin{aligned} d_2(m_0 + \varepsilon m) &= (m_0 + \varepsilon m)\circ\big((m_0+\varepsilon m)\otimes id - id \otimes (m_0 + \varepsilon m)\big) \\ &= \cancel{d(m_0)}^{\,0} + \varepsilon\, m \circ (m_0 \otimes id - id \otimes m_0) \\ &\quad + \varepsilon\, m_0 \circ (m \otimes id - id \otimes m) \end{aligned}$$

This is __linear__ in m

Likewise a 1st order automorphism is $x \in \underset{\substack{\| \\ Hom(\underline{A},A)}}{gl(\underline{A})}$

$$(1+\varepsilon x)\cdot m = m + \varepsilon [x, m]$$

$\Rightarrow$ we have

$$\begin{array}{ccccc} \underline{0} & & \underline{1} & & \underline{2} \\ Hom(A,A) & \xrightarrow{d_1} & Hom(A^{\otimes 2}, A) & \xrightarrow{d_2} & Hom(A^{\otimes 3}, A) \\ x & \longmapsto & [x, m_0] & & \end{array}$$

$$H^1 = \text{1st order defs} \Big/ \text{1st order automorphisms}$$

Explicit examples

II

① Grouping of C_2 : $\mathbb{C}[x]/(x^2=1)$

$$m_0(1,1)=1 \quad , m_0(1,x)=x=m_0(x,1) \quad , m_0(x,x)=1$$

$$d_{2,m_0}(m) = m_0\circ(m\otimes id - id\otimes m) + m\circ(m_0\otimes id - id\otimes m_0)$$

Study m:

$1,1,1$: $\underbrace{m(1,1)-m(1,1)}_{0} + \underbrace{m_0(1,1)-m(1,1)}_{0}=0$

$1,1,x$: $m(1,1)\cdot x - m(1,x) + \underbrace{m(1,x)-m(1,x)}_{0}=0$

$$m(1,x) := x\cdot m(1,1)$$

$1,x,x$: $m(1,x)\cdot x - \underbrace{m(x,x) + m(x,x)}_{0} - m(1,1)=0$

$$m(1,1) = x\cdot m(1,x)$$

x,x,x : $\underbrace{x\cdot m(x,x) - x\, m(x,x)}_{0} + \underbrace{m(x,x) - m(x,x)}_{0} = 0$

⋮

$$\underbrace{m(1,x)=m(x,1)=x\cdot m(1,1) \quad , \quad m(x,x) \text{ unconstrained}}_{\text{2-dimensional space of cocycles}}$$

What about coboundaries? $g(1)=a+bx$, $g(x)=c+dx$

then $d_{1,m_0}(g)=[g,m_0]=$
$(1,1)\longmapsto a+bx$
$(1,x)$ & $(x,1) \longmapsto b+ax$
$(x,x)\longmapsto (2d-a)+(2c-b)$

$\Rightarrow H^1 = 0$!

The group ring is "rigid" (no ^unnatural^ deformations), which is reasonable since finite groups are

② dual numbers : $\mathbb{C}[x]/(x^2)$

JUST COPY SZENDROI's

"Def theory" computation

A dg Lie algebra

II

$\mathrm{Hom}(A,A)$ has a Lie algebra structure via the commutator bracket

There's a natural extension of this to 2- and 3- and n-ary operations:

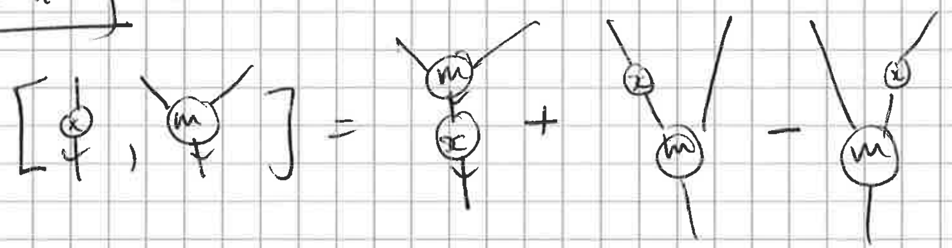
$$f \in \mathrm{Hom}(A^{\otimes m}, A), \quad g \in \mathrm{Hom}(A^{\otimes n}, A)$$

$$[f,g](a_1, \dots, a_{m+n-1}) =$$

$$\sum_{i=1}^{n} (-1)^{in} f(a_1 \dots a_{i-1}\, g(a_i \dots a_{i+n-1})\, a_{i+n} \dots a_{n+m-1})$$

$$+ \sum_{j=1}^{m} (-1)^{jm} g(\text{———}\ f(\text{———})\ \text{———})$$

Graphically:



Hence for $m \in \mathrm{Hom}(A^{\otimes 2}, A)$,

$$[m, -] : \mathrm{Hom}(A^{\otimes n}, A) \longrightarrow \mathrm{Hom}(A^{\otimes n+1}, A)$$

& you can check

$$[m,m] = 0 \iff d_2 m = 0$$

Thus the cochain cplx

$$\begin{array}{ccccccc} \underline{-1} & & \underline{0} & & \underline{1} & & \\ \mathrm{Hom}(k,A) & \xrightarrow{[m_0,-]} & \mathrm{Hom}(A,A) & \xrightarrow{[m_0,-]} & \mathrm{Hom}(A^{\otimes 2},A) & & \cdots \end{array}$$

is a dg Lie algebra

It is known a Hochschild cochains $\overset{\text{up to shift}}{\rightsquigarrow} HH^*(A,A)$

& $\cancel{HH}^{-1} =$ center of $A = \{ a : \underset{a* - \cdots * a}{\overset{\shortparallel}{[m_0, a]}} = 0 \}$

$H^0 =$ derivations of A / "inner" derivations "$[a,-]$"

$H^1 =$ 1^{st} order deformations / 1^{st} order automorphisms

<u>Q</u>: What do other cohomology groups mean?

<u>A</u>: Good question!
We'll give a kind of answer later in the course but the higher groups rarely have a concrete interpretation.
(Consider the analogous question about higher derivatives of a function!)

II

~~Remark~~ | Application

With this example in mind, we can talk about a very interesting result that uses these ideas, known as Kontsevich's formality theorem (or existence of def quantization).

- A classical mechanical system is identified with the mathematical notion of a Poisson manifold (which includes symplectic mflds).
  $\pi \in \Gamma(X, \Lambda^2 T_X)$
- A <u>deformation quantization</u> of a Poisson manifold (X, π) is an ~~bi~~linear associative product $\star$ on $C^\infty(X)[[\hbar]]$ such that

$$\forall \; f = f_0 + \hbar f_1 + \dots + \hbar^n f_n + \dots$$
$$g = g_0 + \hbar f_1 + \dots$$

$$f \star g - g \star f \mod \hbar^2 = \hbar \{f_0, g_0\}$$
$$\{f_0, g_0\} = \pi(df \wedge dg)$$

Rmk: we require $\star$ to be "local" on X

What Kontsevich showed, was that there is an equivalence of deformation functors (in the derived sense)

$$\left\{ \begin{array}{l} \text{def's of} \\ (X, \pi) \text{ as} \\ \underline{\text{Poisson mfld}} \end{array} \right\} \xrightarrow{\simeq} \left\{ \begin{array}{l} \text{"local" defs of} \\ C^\infty(X) \text{ as} \\ \underline{\text{associative}} \text{ algs} \end{array} \right\}$$

$HH^*_{diff}(C^\infty(X), C^\infty(X))$

"classical mechanical systems" — "quantum mechanical systems"

He does this by pinning down the appropriate dg Lie algebras & constructing a weak equivalence of them (via the machinery of $\underline{L_\infty \text{ algebras}}$)

Geometry

Let X be a cplx manifold

G a complex Lie group (think $GL_n\mathbb{C}$)

The following specifies a <u>holomorphic principal</u> <u>G-bundle</u> $P \longrightarrow X$:

- a cover $\{U_i\}$ of X
- patching data $\{g_{ij} : \underset{}{U_{ij}} \longrightarrow G\}$ such that ($U_{ij} = U_i \cap U_j$)

① g_{ij} is holomorphic

$$\begin{tikzcd} & U_{ij} \times G \arrow[dl, "id \times id"'] \arrow[dr, "id \times g_{ij}"] & \\ U_i \times G & & U_j \times G \end{tikzcd}$$

② $g_{ij}\, g_{jk} = g_{ik}$ as functions on U_{ijk}

Recall that $\mathrm{Lie}(G) =: \mathfrak{g}$ describes "1st order neighborhood of $1_G \in G$"

Then a 1st order deformation of this patching data

$$g_{ij} \rightsquigarrow g_{ij}(1_G + \varepsilon \tilde{g}_{ij}) \qquad \boxed{\varepsilon^2 = 0}$$

is ① $\tilde{g}_{ij} : U_{ij} \longrightarrow \mathfrak{g}$ holomorphic

2

$$g_{ik}(1+\varepsilon\tilde{g}_{ik}) = g_{ij}(1+\varepsilon\tilde{g}_{ij})\, g_{jk}(1+\varepsilon\tilde{g}_{jk})$$

$$\Updownarrow \ \varepsilon\text{-term}$$

$$g_{ik}\tilde{g}_{ik} = g_{ij}\tilde{g}_{ij} g_{jk} + \overset{g_{ik}}{\cancel{g_{ij} g_{jk}}}\,\tilde{g}_{jk}$$

$$\Updownarrow \ \text{left multiply by } g_{ik}^{-1}$$

$$\tilde{g}_{ik} = \mathrm{ad}(g_{jk})(\tilde{g}_{ij}) + \tilde{g}_{jk}$$

Let $\mathrm{ad}\,P = P \times^{G}_{\mathrm{ad}} \mathfrak{g} \longrightarrow X$ denote the <u>adjoint bundle</u>

Such data $\{\tilde{g}_{ij}\}$ specifies a Čech 1-cocycle for $\mathrm{ad}\,P$.

By refining covers & taking into account automorphisms of a principal bundle, one finds

$$\boxed{H^1(X, \mathrm{ad}\,P) \cong 1^{st} \text{ order defs} \,/\, 1^{st} \text{ order automorphisms}}$$

Let $\mathcal{A}d_P$ denote the sheaf of holomorphic sections of $\mathrm{ad}\,P$. As $\mathfrak{g}$ is a Lie algebra, $\mathcal{A}d_P(U)$ is also a Lie algebra for every open $U \subset X$.

If we "describe" it via its Dolbeault complex

$$\Omega^{0,*}(X, \mathrm{ad}\,P)$$

we really have a dg Lie algebra, fitting w/ the theme of the course.

Lets move examples: character varieties, instanton spaces, ...